J. Fluid Mech. (1968), *vol.* 31, *part* 1, *pp.* 193–205
Printed in Great Britain

The diffraction of Kelvin waves at a corner

By V. T. BUCHWALD
Department of Applied Mathematics, The University of Sydney

(Received 23 March 1967)

In a uniform rotating liquid of uniform depth, Kelvin waves may be propagated in one direction along a straight boundary of the liquid. The Wiener–Hopf technique is used to obtain the wave field due to Kelvin waves incident at a right-angle corner. An asymptotic solution is obtained for the field far from the corner. It is shown that for low frequencies the Kelvin waves are propagated round the corner without change of amplitude, but that for high frequencies cylindrical waves of the 'Poincaré' type are generated at the corner, so that the amplitudes of the Kelvin waves propagated round the corner are reduced.

1. Introduction

Kelvin waves in a semi-infinite canal have been considered in a paper by Taylor (1921), in which it is concluded that reflexion of the Kelvin wave at the transverse wall is only possible if the width of the canal is less than a certain value, which depends on the frequency. Defant (1961) has given an alternative presentation of Taylor's solution. In both cases the solution is in the form of a Fourier series, the coefficients of which can only be evaluated by the approximate inversion of an infinite matrix.

In view of some of the questions raised by the above analyses, it is of some theoretical interest to know what happens to a Kelvin wave when it reaches a right-angle bend in a straight coast line. In this paper, the device of extending the variables to the whole half-plane is used to reduce the problem to the solution of a Wiener–Hopf equation. A closed solution is obtained in the form of a Fourier integral and expressions are obtained for the amplitudes of the reflected waves at a long distance from the corner. It will be shown that, depending on the frequency, the Kelvin waves are either propagated round the bend without change of amplitude, or, alternatively, they are propagated with reduced amplitude, but with the addition of cylindrical waves of the 'Poincaré' type which radiate from the corner.

2. The equations of motion and boundary conditions

Assuming a time factor $e^{i\omega t}$, the linearized equations of motion of long waves in a sheet of water of uniform depth h, rotating about a vertical axis with angular velocity $\frac{1}{2}f$, are, in rectangular Cartesian co-ordinates,

$$\left.\begin{aligned} i\omega u - fv &= -g\zeta_x, \\ i\omega v + fu &= -g\zeta_y, \\ h(u_x + v_y) + i\omega\zeta &= 0. \end{aligned}\right\} \tag{2.1}$$

In these equations $u(x,y)$, $v(x,y)$ are the particle velocities, averaged over the depth, in the x, y, directions, respectively, $h+\zeta$ is the depth in the disturbed state, and the suffices indicate partial derivatives in the usual way. In the case of the rotating earth, $f = 2\Omega \sin\theta$, where θ is the latitude, but for the purpose of this paper it is reasonable to assume that f is constant. It is also assumed that ω has small negative imaginary part, so that

$$\omega = \sigma - i\epsilon, \tag{2.2}$$

where $0 < \epsilon \ll \sigma$. This is a common and convenient way in which the radiation condition can be applied. The steady-state solution can be obtained afterwards by taking ϵ to be zero.

Eliminating u, v, respectively from the first two equations of (2.1),

$$\left.\begin{aligned} hk^2 u &= i\omega\zeta_x + f\zeta_y, \\ hk^2 v &= -f\zeta_x + i\omega\zeta_y, \end{aligned}\right\} \tag{2.3}$$

where $k^2 = (\omega^2 - f^2)/c_0^2$, and $c_0 = \surd(gh)$ is the velocity of long waves when $f = 0$. Using the last of (2.1),

$$(\nabla^2 + k^2)\,\zeta = 0, \tag{2.4}$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. The velocities u, v, also satisfy this equation.

Now suppose that the water is contained by smooth impenetrable barriers in the positive quadrant $x > 0$, $y > 0$, so that $u(0,y) = 0$, and $v(x,0) = 0$ are the conditions on the boundaries $x = 0$ and $y = 0$. Then, as given by Proudman (1953),

$$\zeta_0 = A \exp\left[(-fx + i\omega y)/c_0\right] \tag{2.5}$$

represents a Kelvin wave of 'amplitude' $|A|$, travelling in the negative direction along the y-axis. Note that ζ_0 satisfies (2.4), and that if u_0, v_0, are the corresponding particle velocities calculated from (2.3), then

$$u_0 \equiv 0, \quad v_0 = -g\zeta_0/c_0, \tag{2.6}$$

so that the condition on $y = 0$ is satisfied. If, therefore, ζ_0 is taken as the incident Kelvin wave, then we need to find the diffracted displacement ζ_1, with corresponding velocities, u_1, v_1, such that

$$(a) \quad u_1 = 0 \qquad \text{at} \quad x = 0, y \geqslant 0, \tag{2.7}$$

and

$$(b) \quad v_0 + v_1 = 0 \quad \text{at} \quad y = 0, x \geqslant 0. \tag{2.8}$$

From (2.7) and (2.4), $\partial^{2n}u_1/\partial x^{2n} = 0$, at $x = 0$, $y > 0$, for any integer n. Hence u_1 is an odd function of x, and is analytic and not exponentially unbounded for all values of x in the half-plane $y > 0$. It may be seen from the first of (2.3) that $\zeta_1 = \alpha\zeta_0 + \zeta^*$, where ζ^* is analytic and not exponentially unbounded for $y > 0$, ζ_0 is given in (2.5), and α is a constant. Application of the radiation condition excludes ζ_0, so that the diffracted wave ζ_1 is an analytic function of x, y, in the half-plane $y > 0$, and is not exponentially large anywhere in this domain.

3. The diffracted wave

Let the diffracted wave be given in the form

$$\zeta_1(x,y) = \int_{-\infty}^{\infty} \Phi(\beta)\, e^{-i\beta x - sy}\, d\beta \quad (y > 0). \tag{3.1}$$

The above expression satisfies the differential equation (2.4) if

$$s = (\beta^2 - k^2)^{\frac{1}{2}}. \tag{3.2}$$

The function $s(\beta)$ given in (3.2) is two-valued, with branch points at $\beta = \pm k$. For convergence of the integral in (3.1), it is necessary to choose a branch such that $s \sim |\beta|$, as $\beta \to \infty$. If $\sigma > f$ the branch points and cuts are chosen as in figure 1*a*, and the chosen branch is the one for which $s = ik$ when $\beta = 0$. In the case

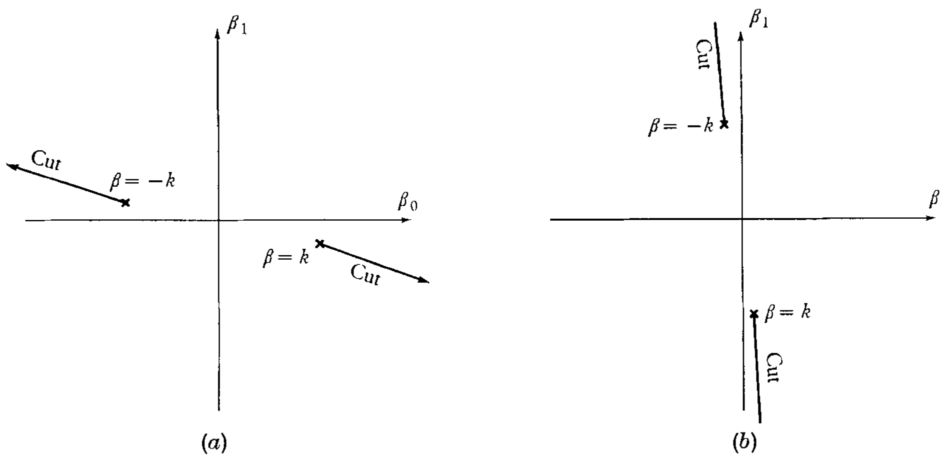

FIGURE 1. Diagram of cuts in the β-plane, where $\beta = \beta_0 + \beta_1$, for the cases (*a*) $\sigma > f$, and (*b*) $\sigma < f$, respectively.

$\sigma < f$, illustrated in figure 1*b*, write $k = ik_1$, where $c_0 k_1 = (f^2 - \omega^2)^{\frac{1}{2}}$, so that the chosen branch is the one for which $s = k_1$ at $\beta = 0$.

From (2.3) and (3.1) it may be seen that the boundary condition (2.7) is satisfied if

$$\int_{-\infty}^{\infty} (\omega\beta - fs)\, \Phi(\beta)\, e^{-sy}\, d\beta = 0.$$

Replacing β by $-\beta$ in the interval $(-\infty, 0)$ this equation may be written in the form

$$\int_{0}^{\infty} [(\omega\beta - fs)\, \Phi(\beta) - (\omega\beta + fs)\, \Phi(-\beta)]\, e^{-sy}\, d\beta = 0,$$

so that a sufficient condition for the satisfaction of the boundary condition (2.7) is

$$(\omega\beta - fs)\, \Phi(\beta) = (\omega\beta + fs)\, \Phi(-\beta). \tag{3.3}$$

To satisfy the other boundary condition (2.8), let

$$v_1(x, 0) = \int_{-\infty}^{\infty} V(\beta)\, e^{-i\beta x}\, d\beta, \tag{3.4}$$

where $$V(\beta) = V^+(\beta) + V^-(\beta),$$

and $$2\pi V^+(\beta) = \int_0^\infty v_1(x,0)\, e^{i\beta x}\, dx,$$

$$2\pi V^-(\beta) = \int_{-\infty}^0 v_1(x,0)\, e^{i\beta x}\, dx.$$

From (2.8), (2.6) and (2.5), the boundary condition (2.8) is satisfied for $x > 0$ if

$$V^+(\beta) = B/(\beta + id), \tag{3.5}$$

where $B = igA/2\pi c_0$ and $d = f/c_0$. In defining $V^-(\beta)$ as above, the definition of $v_1(x,0)$ has been extended to negative values of x. The function $V^-(\beta)$ has yet to be determined, but assuming that $v_1(x,0) = O(e^{\delta x})$, as $x \to -\infty$, where $\delta > 0$

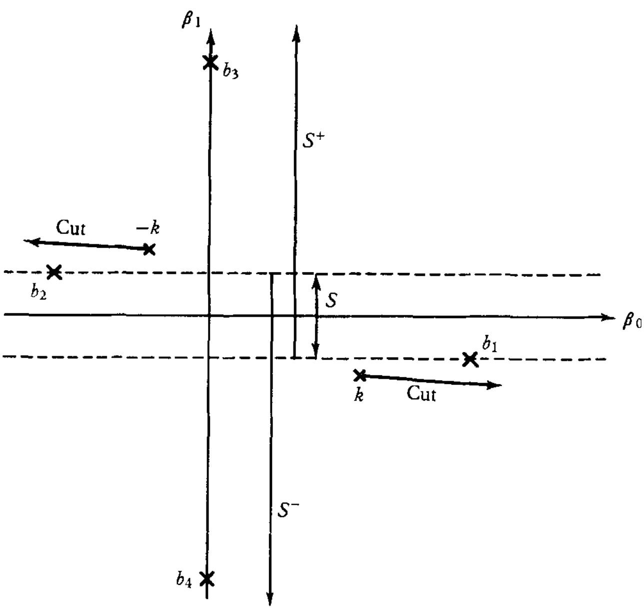

FIGURE 2. The singularities and strips of analyticity in the β-plane in the case $\sigma > f$. The case $\sigma < f$ is the same, except that the singularities at $\beta = \pm k$, and the cuts are as in figure 1 b.

is a constant to be found, $V^-(\beta)$ is analytic for β in the half-plane S^-, illustrated in figure 2. If $\beta = \beta_0 + i\beta_1$, then S^- is the half-plane $\beta_1 < \delta$. It will be shown below that $\delta = \epsilon/c_0$.

Assuming that (3.1) defines ζ_1 in the whole half-plane $y \geqslant 0$, then, from (2.3),

$$hk^2 v_1(x,y) = i\int_{-\infty}^{\infty} (f\beta - \omega s)\, \Phi(\beta)\, e^{-i\beta x - sy}\, d\beta.$$

Hence, putting $y = 0$ and comparing the integrand with that in (3.4), the relation

$$hk^2\, V(\beta) = i(f\beta - \omega s)\, \Phi(\beta) \tag{3.6}$$

is obtained. Substitution of (3.6) and (3.5) in (3.3) yields

$$V^-(\beta) + \frac{B}{\beta+id} + G(\beta)\left[V^-(-\beta) - \frac{B}{\beta-id}\right] = 0, \tag{3.7}$$

where

$$G(\beta) = \frac{(\omega\beta+fs)(f\beta-\omega s)}{(\omega\beta-fs)(f\beta+\omega s)}. \tag{3.8}$$

Using the notation

$$b_1 = -b_2 = \omega/c_0, \quad b_3 = -b_4 = id = if/c_0,$$

it can be shown that $G(\beta)$ has simple zeros at $\beta = b_2$, $\beta = b_3$, and simple poles at $\beta = b_1$, $\beta = b_4$. There are also branch points at $\beta = \pm k$, with cuts as in figure 1. With these singularities in mind, noting that $|\mathscr{I}(k)| > \epsilon/c_0$, and making the assumption that $\epsilon < f$, it may be inferred from (3.7) that $\delta = \epsilon/c_0$, so that S^- is the half-plane $\beta_1 < \epsilon/c_0$. If S^+ denotes the half-plane $\beta_1 > -\epsilon/c_0$, and $S = S^+ \cap S^-$ is the strip $|\beta_1| < \epsilon/c_0$, then $V^-(-\beta)$ is analytic for β in S^+, and equation (3.7) holds for β in the strip S. [The various singularities and regions are illustrated in figure 2 for the case $\sigma > f$.]

Equation (3.7) is of the Wiener–Hopf type, and the function $G(\beta)$ can be factorized in the form

$$G(\beta) = G^+(\beta)/G^-(\beta), \tag{3.9}$$

where $G^+(\beta)$ is analytic and non-zero for β in S^+, $G^-(\beta)$ is analytic and non-zero for β in S^-, and

$$G^-(\beta) = (\beta-b_2)(\beta-b_3)\prod_{j=1}^{4}\Psi_j(\beta), \tag{3.10}$$

where

$$\Psi_j(\beta) = \exp\left\{-\frac{f\omega}{c_0^2}\int_{\beta^*}^{\beta} q_j(\tau)\,[\psi(\tau)-\psi(b_j)]\,d\tau\right\}, \tag{3.11}$$

with

$$q_j(\tau) = b_j^{-1}(\tau-b_j)^{-1},$$

and

$$\psi(\tau) = (\pi s)^{-1}\cos^{-1}(\tau/k), \tag{3.12}$$

where $s = (\tau^2-k^2)^{\frac{1}{2}}$, and β^* is an arbitrary constant. The function $\psi(\beta)$ is multi-valued, but if we choose the branch for which $\psi(0) = 1/2ki$, then $\psi(\beta)$ is analytic for β in S^-, its only singularity being a branch point at $\beta = -k$, with an appropriate cut as in figure 1. Some important particular values of $\psi(\beta)$ are given in table 1.

β	(i) $\omega > f$ $\psi(\beta)$	(ii) $\omega < f$ $\psi(\beta)$
0	$1/2ik$	$1/2k_1$
ω/c_0	$-(ic_0/\pi f)\cosh^{-1}(\omega/c_0k)$	$(c_0/2\pi f)[\pi - 2i\sinh^{-1}(\omega/c_0k_1)]$
$-\omega/c_0$	$(c_0/\pi f)[\pi + i\cosh^{-1}(\omega/c_0k)]$	$(c_0/2\pi f)[\pi + 2i\sinh^{-1}(\omega/c_0k_1)]$
if/c_0	$-(c_0/2\pi\omega)[2\sinh^{-1}(f/c_0k) + i\pi]$	$-(c_0/\pi\omega)[\cosh^{-1}(f/c_0k_1) + i\pi]$
$-if/c_0$	$(c_0/2\pi\omega)[2\sinh^{-1}(f/c_0k) - i\pi]$	$(c_0/\pi\omega)\cosh^{-1}(f/c_0k_1)$

TABLE 1. Some particular values of $\psi(\beta)$. In column (i), $\cosh^{-1}(\omega/c_0k) = \sinh^{-1}(f/c_0k)$, and in column (ii) $c_0k_1 = (f^2-\omega^2)^{\frac{1}{2}}$, so that $\cosh^{-1}(f/c_0k_1) = \sinh^{-1}(\omega/c_0k_1)$.

The actual derivation of (3.10) is discussed in appendix A in which some of the properties of $G^-(\beta)$ are also discussed. Two of the properties which should be noted at this stage are

$$G^+(\beta) = G^-(-\beta), \tag{3.13}$$

and $$G^-(\beta) = C\beta + O(1), \tag{3.14}$$
as $\beta \to \infty$, where C is constant.

Equation (3.7) can now be rewritten in the form

$$G^-(\beta)\, V^-(\beta) + \frac{B}{\beta+id}[G^-(\beta) - G^-(-id)] - \frac{B}{\beta-id}G^+(id)$$
$$= -G^+(\beta)\, V^-(-\beta) - \frac{B}{\beta+id}G^-(-id) + \frac{B}{\beta-id}[G^+(\beta) - G^+(id)]. \tag{3.15}$$

The left-hand side of (3.15) is analytic for β in S^-, the right-hand side is analytic for β in S^+, and the equation itself holds for β in the strip S. The condition that $v_1(0,0)$ is finite implies that $V^-(\beta) = O(\beta^{-1})$ as $\beta \to \infty$, so that, using (3.14), it is easily seen that both sides of (3.15) are $O(1)$ as $\beta \to \infty$. Hence, by the usual arguments based on Liouville's theorem,

$$V^-(\beta) = -\frac{B}{\beta+id}\left[1 - \frac{G^-(-id)}{G^-(\beta)}\right] + \frac{B}{\beta-id}\frac{G^+(id)}{G^-(\beta)} + \frac{D}{G^-(\beta)}, \tag{3.16}$$

where D is an arbitrary constant. So far, $v_1(x,0)$ has only been determined for $x > 0$. Now
$$v_1(0,0) = \int_{-\infty}^{\infty} V(\beta)\,d\beta,$$

where $V(\beta)$ is obtained by taking the sum of (3.5) and (3.16). Replacing the integration by a semi-circular contour of large radius R, together with the negative residue at the pole $\beta = id$, it can be shown that as $R \to \infty$,

$$v_1(0,0) = -gA/c_0 + \pi Di/C + O(R^{-1}),$$

so that $D = 0$ if the boundary condition at the origin is to be satisfied. Hence, noting that $G^+(id) = G^-(-id)$ and using (3.1), (3.5) and (3.6),

$$\pi\zeta_1(x,y) = Ac_0k^2G^-(-id)\int_\Gamma \frac{\exp(-i\beta x - sy)}{(f\beta-\omega s)(\beta^2+d^2)\,G^-(\beta)}\,\beta\,d\beta, \tag{3.17}$$

where Γ is the real axis from $-\infty$ to ∞ if $\epsilon > 0$. In the limit, as $z \to 0$, Γ must steer around the singularities as in figure 3 in the case $\omega > f$. A similar contour is also taken in the case $\omega < f$.

Since $s \sim |\beta|$, as $\beta \to \infty$, the integral in (3.17) is uniformly convergent for all positive values of y. Moreover, using (3.14), the integrand is $O(\beta^{-3})$ when $y = 0$, as $\beta \to \infty$. Hence $\zeta_1(x,y)$ is an analytic function of x, y, for all $y > 0$, and has continuous first derivatives with respect to x on $y = 0$. There is a singularity at the origin, where the second derivatives of ζ_1 are discontinuous. Substitution in (2.4), (2.7) and (2.8) shows that $\zeta_1(x,y)$, as given in (3.17), satisfies the differential equation and boundary conditions, differentiation under the integral sign being justified by the convergence of the integrals. Since the radiation condition is also satisfied, (3.17) necessarily gives the unique solution of the problem posed.

4. The asymptotic solution

For given x, y it is possible in principle to obtain $\zeta_1(x,y)$ by quadrature from (3.17). However, a general idea of the nature of the diffracted waves may be obtained by considering the asymptotic form of ζ_1, for large $r = (x^2+y^2)^{\frac{1}{2}}$. Let $x = r\cos\phi$, $y = r\sin\phi$, so that

$$\pi\zeta_1(r,\phi) = Ak^2c_0 G^-(-id)\int_\Gamma \frac{\exp[r\Lambda(\beta)]}{(f\beta-\omega s)(\beta^2+d^2)G^-(\beta)}\beta\,d\beta, \tag{4.1}$$

where
$$\Lambda(\beta) = -i\beta\cos\phi - s\sin\phi. \tag{4.2}$$

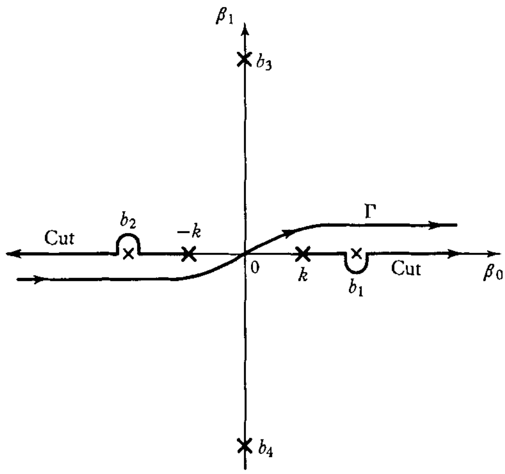

FIGURE 3. Sketch of the singularities and the contour Γ in the β-plane, in the limit as $\epsilon \to 0$, in the case $\sigma > f$. In the case $\sigma < f$, the cuts are as in figure 1 b, but along the imaginary axis when $\epsilon = 0$.

The constant ϵ has now served its purpose in determining the contour Γ, and for the remainder of the paper we take $\epsilon = 0$, and the contour Γ as in figure 3. Now ω is real, and assume, first, that $\omega > f$. Consider the transformation

$$\beta = k\cos\alpha \tag{4.3}$$

which, for $\omega > f$, maps the cut β-plane and contour Γ, illustrated in figure 3, on to the strip $0 < \mathscr{R}(\alpha) < \pi$, and the image contour Γ_1, illustrated in figure 4. In the latter figure, the points B_j ($j = 1, 2, 3, 4$), are the images of the points $\beta = b_j$ respectively, while K_1, K_2, are the images of $\beta = k$, $-k$, respectively. The image of the origin O in the β-plane is at O_1, where $\alpha = \frac{1}{2}\pi$.

The transformed exponent in (4.1) is

$$\Lambda_1(\alpha) = \Lambda(k\cos\alpha) = -ik\cos(\alpha-\phi), \tag{4.4}$$

so that a saddle-point of $\Lambda_1(\alpha)$ is given by

$$\Lambda_1'(\alpha) = ik\sin(\alpha-\phi) = 0,$$

when
$$\alpha = \phi. \tag{4.5}$$

Noting that $\Lambda_1''(\phi) = ik$, it is easily seen that the path of steepest descent through the saddle-point makes an angle of $\frac{5}{4}\pi$ with the real axis. If $\alpha = \alpha_0 + i\alpha_1$, the asymptotes of the path of steepest descent are the lines $\alpha_0 = \phi \pm \frac{1}{2}\pi$, so that the path of steepest descent has the form of Γ^* in figure 4. By the standard formula used in the method of steepest descent (see, for instance, Jeffreys & Jeffreys 1956), the contribution from the saddle-point to the integral is

$$\zeta_{11} = Ak^2c_0\left\{\frac{k}{2\pi r}\right\}^{\frac{1}{2}} \frac{G^-(-id)\exp\left[-i(kr-\frac{1}{4}\pi)\right]\sin 2\phi}{G^-(k\cos\phi)\,(f\cos\phi - i\omega\sin\phi)\,(d^2+k^2\cos^2\phi)} + O[(kr)^{-1}]. \quad (4.6)$$

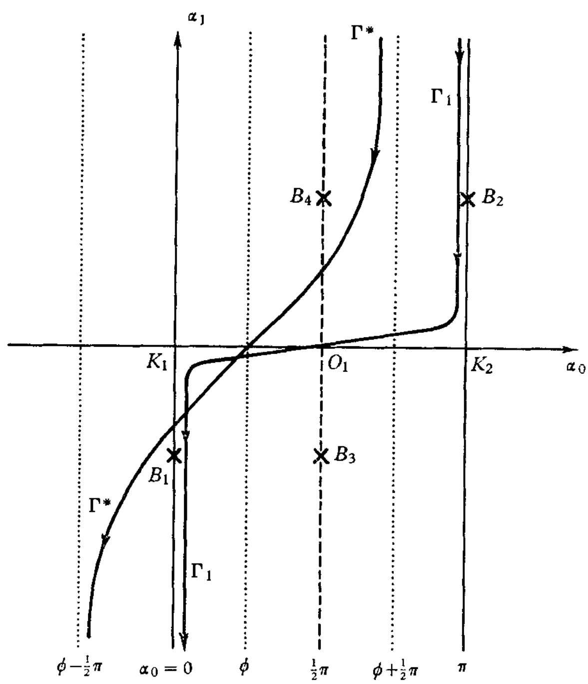

FIGURE 4. Image of figure 3 in the α-plane, where $\beta = k\cos\alpha$, and $\omega > f$. Γ_1 is the image of Γ, while Γ^* is the path of steepest descent.

When $\phi = \frac{1}{2}\pi$, the saddle-point is at O_1, and Γ_1 may be deformed into Γ^* without capturing any of the poles, so that (4.6) gives the complete asymptotic solution. As ϕ decreases from $\frac{1}{2}\pi$, first the pole at B_1, and then the pole at B_4 are captured, so that contributions from the residues at both these points must be taken into account. However, as can be seen from figure 4, the pole at B_4 is only captured for small values of ϕ, and by then the contribution from this pole is exponentially small compared with ζ_{11}. The contribution from the pole at B_1 is $-2\pi i$ times the residue at B_1, which, when simplified, yields

$$\zeta_{12} = 2iA\frac{wf}{\omega^2+f^2}\frac{G^-(-if/c_0)}{G^-(\omega/c_0)}\exp\left(-i\omega x/c_0 - fy/c_0\right). \quad (4.7)$$

For most values of ϕ, ζ_{11} will give the asymptotically dominant displacement. However, noting that $\zeta_{11} \sim r^{-\frac{1}{2}}$, and $\zeta_{11} = 0$ when $\phi = 0$, it is ζ_{12} which is important in the neighbourhood of the x-axis, especially as ζ_{12} does not diminish with distance in the x-direction.

There is a simple physical explanation for the results in (4.6) and (4.7). The displacement ζ_{12} is a reflected Kelvin wave travelling in the positive direction along the x-axis with speed c_0. On the other hand, ζ_{11} is a cylindrical Poincaré type wave radiating from the origin, and travelling with a phase velocity ω/k. Whilst not a consequence of the monochromatic analysis used in this paper, it should be noted that the group velocity of the Poincaré waves is kc_0^2/ω, and this would have

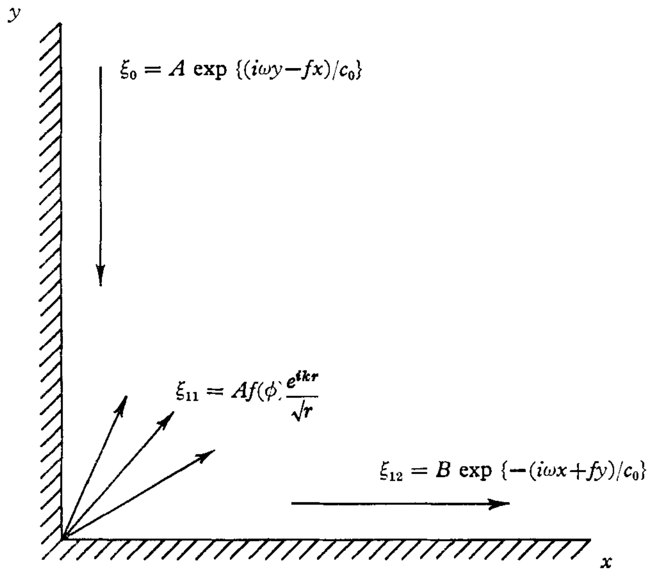

FIGURE 5. Illustrates the incident Kelvin wave ζ_0, the reflected Kelvin wave ζ_{12}, and the diffracted cylindrical wave ζ_{11} of the Poincaré type.

to be taken into account in any discussion of the energy associated with these waves. Figure 5 illustrates the physical features associated with the asymptotic solution.

The procedure in the case $\omega < f$ is much the same as the one above. If $k = -ik_1$, where $k_1 = (f^2 - \omega^2)^{\frac{1}{2}}$, and $s = (\beta^2 + k_1^2)^{\frac{1}{2}}$, the transformation (4.3) maps the β-plane, cut as in figure 1*b*, on to the strip $0 < \omega_0 < \pi$, illustrated in figure 6. The contour Γ_1, the image of Γ, nearly coincides with the line $\alpha_0 = \frac{1}{2}\pi$. The saddle-point is at $\alpha = \phi$, as before, but the path of steepest descent Γ^* is, as illustrated, parallel to the imaginary axis. Without going into the actual details, it is evident that now

$$\zeta_{11} \sim r^{-\frac{1}{2}} e^{-k_1 r}. \tag{4.8}$$

In the deformation of the contour from Γ_1 to Γ^*, the pole at B_1 is captured for almost all values of ϕ, while B_4 can be captured only for $\phi = 0$, when its contribution is exponentially small compared with the contribution from B_1. It is not difficult to show that the formula (4.7) for ζ_{12} also applies in this case, so that for large distances from the corner there exist only the reflected Kelvin waves travelling in the positive direction along the x-axis.

Arguing on physical grounds, in the case $\omega < f$ the diffracted Kelvin waves must have the same amplitudes as the incident Kelvin waves. It can be shown analytically, after some computation, that in this instance $|\zeta_{12}| = |A|$, confirming that for $\omega < f$ the incident Kelvin waves travel round the corner without changing amplitudes. Some details of this computation are given in appendix B.

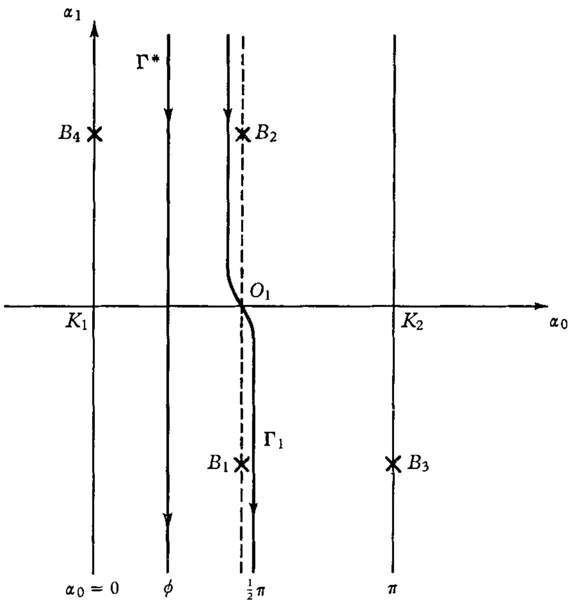

FIGURE 6. Image of figure 3 in the α-plane, in the case $\omega < f$, so that $\beta = ik_1 \cos\alpha$. The path of steepest descent is denoted by Γ^*.

5. Conclusions

The above calculations have been made for positive f only, that is for a system which rotates in the sense ϕ increasing. For rotations in the opposite sense, f would be negative, and the incident Kelvin wave would have to travel in the negative direction along the x-axis, the reflected wave being in the positive y-direction. Thus the above results can be extended to negative values of f by reflexion in the line $y = x$.

In his paper, Taylor (1921) concluded that for fixed $\omega > f$, Kelvin waves can only be reflected in a semi-infinite canal if the width is less than a certain critical value. It would appear from the above analysis that since the corners act as sources of Poincaré waves, and because the canal can act as a wave guide for these waves, the critical width discovered by Taylor must be the 'cut-off' width for the canal to act as a wave guide in this way. One would therefore expect the Kelvin waves to be totally reflected if the canal width is less than the 'cut-off' width. However, for greater widths the energy of the incident Kelvin waves will be distributed among the reflected Kelvin waves, and all the possible modes of the reflected Poincaré waves guided along the canal.

From the theoretical point of view the combination of Fourier transforms with the Wiener–Hopf procedure has proved to be a comparatively simple technique for solving the particular problem posed in this paper, especially as only elementary functions are used. Examples of other techniques which have been used to solve related problems are given by Karp & Karal (1959), Karal, Karp, Chu & Kouyoumjian (1961) and Maluzhinets (1958).

This work was finished during leave of absence which was supported in part by a Royal Society and Nuffield Foundation Commonwealth Bursary. I am indebted to Mr J. Crease for some helpful discussions, and to the staff of the National Institute of Oceanography, England, for preparation of the typescript and figures.

Appendix A

Functions of the form $G(\beta)$, given in (3.8), have been factorized by Heins & Feshbach (1954), Senior (1952), and are discussed by Noble (1958). Let

$$H(\beta) = \frac{d}{d\beta}\log G(\beta), \tag{A 1}$$

then

$$H(\beta) = \frac{f\omega}{sc_0^2}\sum_{j=1}^{4} q_j(\beta), \tag{A 2}$$

in the notation of (3.11). The branch of the function

$$\psi(\beta) = (\pi s)^{-1}\cos^{-1}(\beta/k) \tag{A 3}$$

for which $\psi(0) = 1/2ki$ has a singularity at $\beta = -k$. With a suitable cut, $\psi(\beta)$ is analytic for β in S^-, so that $\psi(-\beta)$ is analytic for β in S^+, and the sum has the property that

$$\psi(\beta)+\psi(-\beta) = s^{-1}. \tag{A 4}$$

Hence it is a simple matter to express $H(\beta)$ in the form

$$H(\beta) = H^+(\beta)+H^-(\beta),$$

where

$$H^-(\beta) = H^+(-\beta) = \frac{f\omega}{c_0^2}\sum_{j=1}^{4}\{q_j(\beta)[\psi(\beta)-\psi(b_j)]\}-(\beta-b_2)^{-1}-(\beta-b_3)^{-1}. \tag{A 5}$$

Equations (3.10) to (3.13) follow immediately on integrating and taking the exponent. The lower limit of integration β^* is arbitrary, giving an arbitrary multiplicative constant in $G^-(\beta)$. However, this constant cancels in (3.9) and (3.17), and is, therefore, immaterial.

To obtain the asymptotic form of $G^-(\beta)$, consider the behaviour of $H^-(\beta)$, as $\beta\to\infty$. Noting that $\psi(\beta) = O(\beta^{-1}\log\beta)$, it is easily seen that

$$H^-(\beta) = -\frac{2}{\beta}-\frac{f\omega}{c_0^2\beta}\sum_{j=1}^{4}[\psi(b_j)/b_j]+o(\beta^{-1}),$$

as $\beta\to\infty$. Calculating values of $\psi(b_j)$ is rather complicated; the results of the calculations are given in table 1. Moreover, $\psi(b_j)$ takes different values, depending on whether k^2 is positive or negative, but, nevertheless, in both cases

$$\sum_{j=1}^{4} [\psi(b_j)/b_j] = -c_0^2/f\omega,$$

so that
$$H^-(\beta) = -\beta^{-1}+o(\beta^{-1}), \tag{A 6}$$

as $\beta \to \infty$. Integrating and taking the exponent of (A 6) gives (3.14), the actual magnitude of C depending on the lower limit of integration.

Appendix B

In order to evaluate $|\zeta_{12}|$ in the case $\omega < f$, first consider

$$Z = Z_1 - Z_2, \tag{B 1}$$

where
$$Z_1 = \mathscr{R}\left\{\int_0^{-if/c_0} \sum_{j=1}^{4} q_j(\tau)\,[\psi(\tau)-\psi(b_j)]\,d\tau\right\}, \tag{B 2}$$

$$Z_2 = \mathscr{R}\left\{\int_0^{\omega/c_0} \sum_{j=1}^{4} q_j(\tau)\,[\psi(\tau)-\psi(b_j)]\,d\tau\right\}, \tag{B 3}$$

the notation being as in (3.11), with the paths of integration along the negative imaginary axis for (B 2) and the positive real axis for (B 3).

If τ is real, $q_1(\tau)$ and $q_2(\tau)$ are real, while $q_3(\tau)$ and $q_4(\tau)$ are complex and conjugate. Also, for real τ and $\omega < f$,

$$\psi(\tau) = \tfrac{1}{2}s^{-1} - (i/\pi s)\sinh^{-1}(\tau/k_1),$$

where k_1, and $s = (\tau^2+k_1^2)^{\frac{1}{2}}$ are real. Also, from table 1,

$$\mathscr{R}[\psi(b_1)] = \mathscr{R}[\psi(b_2)] = c_0/2f,$$

and
$$\mathscr{R}[q_3(\tau)\,\psi(b_3)+q_4(\tau)\,\psi(b_4)] = -\frac{c_0^2\,\tau}{\omega f(\tau^2+d^2)}.$$

Hence, recalling formula (A 2) for $H(\beta)$ and (3.8) for $G(\beta)$, and noting that

$$(f\beta-\omega s)/(\tau-\omega/c_0) \to f-\omega^2/f \quad \text{as} \quad \tau \to \omega/c_0,$$

we find that
$$\frac{f\omega}{c_0^2}Z_2 = \frac{1}{2}\int_0^{\omega/c_0}\left\{H(\tau)+\frac{1}{\tau+\omega/c_0}-\frac{1}{\tau-\omega/c_0}-\frac{\tau}{\tau^2+d^2}\right\}d\tau$$
$$= \tfrac{1}{2}\left[\log\left[(\tau^2+d^2)\,(\tau+\omega/c_0)\,G(\tau)/(\tau-\omega/c_0)\right]\right]_0^{\omega/c_0}$$
$$= \log\left[(\omega^2+f^2)/f^2\right]. \tag{B 4}$$

If we consider figure 6, we see that on O_1K_1, both s and $\cos^{-1}(\tau/k)$ are real, while on the segment K_1B_4 they are both imaginary. Hence $\psi(\tau)$ is purely real on the path of integration in (B 2). If we put $\tau = -i\lambda$, then $\sum_{j=1}^{4} q_j(-i\lambda)$ is purely imaginary. From table 1, $\psi(b_1)$ and $\psi(b_2)$ are complex and conjugate, $\psi(b_4)$ is real, and the imaginary part of $\psi(b_3)$ is $-ic_0/\omega$. Consequently

$$\frac{f\omega}{c_0^2}Z_1 = \int_0^d \frac{d\lambda}{\lambda+d} = \log 2. \tag{B 5}$$

From (4.7), (3.10), (3.11), (B 1), (B 2) and (B 3), when $\omega < f$,

$$|\zeta_{12}| = \frac{2f^2}{\omega^2+f^2}|A|\exp\left[-\frac{f\omega}{c_0^2}(Z_1-Z_2)\right] = |A|. \tag{B 6}$$

REFERENCES

Defant, A. 1961 *Physical Oceanography*, vol. II, pp. 202–217. Oxford: Pergamon Press.

Heins, A. E. & Feshbach, H. 1954 *Proc. Symp. Appl. Math.* **5**, 75.

Jeffreys, H. & Jeffreys, B. S. 1956 *Methods of Mathematical Physics*, Chapter 17. Cambridge University Press.

Karal, F. C., Karp, S. N., Chu, T. S. & Kouyoumjian, R. G. 1961 *Comm. Pure Appl. Math.* **14**, 35.

Karp, S. N. & Karal, F. C. 1959 *Comm. Pure Appl. Math.* **12**, 435.

Maluzhinets, G. D. 1958 *Soviet Phys. Dokl.* **3**, 752.

Noble, B. 1958 *Methods Based on the Wiener–Hopf Technique*. Oxford: Pergamon Press.

Proudman, J. 1953 *Dynamical Oceanography*. London: Methuen.

Senior, T. B. A. 1952 *Proc. Roy. Soc.* A **213**, 436.

Taylor, G. I. 1921 *Proc. Lond. Math. Soc.* **20**, 148.